Modulated phases in magnetic models frustrated by long-range interactions

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We study an Ising model in one dimension with short-range ferromagnetic and long-range (power law) antiferromagnetic interactions. We show that the zero temperature phase diagram in a (longitudinal) field H involves a sequence of up and down domains whose size varies continuously with H, between $-H_c$ and H_c, which represent the edges of the ferromagnetic up and down phases. The implications of long-range interaction in many body systems are discussed.

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I. INTRODUCTION

It is known that long-range interactions lead to significant changes in the behavior of interacting many body systems. Thus, though the Ising model in one dimension with a short-range ferromagnetic interaction does not exhibit a phase transition, the corresponding model with a ferromagnetic interaction that falls off as a power law $1/r^{\lambda}$ has a phase transition at nonzero temperature[1–3] for $\lambda \leq 2$. A similar effect is found for the Ising spin glass in one dimension, which is unfrustrated with nearest neighbor interactions, but becomes sufficiently frustrated with a power-law interaction so that for $\lambda < 1$ a finite T transition is attained.[4,5]

For short-range models with a finite transition temperature, addition of long-range interactions having a power-law falloff can lead to changes in the universality class of the phase transition. For sufficiently small power-law exponent, critical exponents are found to vary continuously with the power-law exponent.[6]

There have been several studies of the Ising model in higher dimensions, frustrated by Coulomb[7,8] or dipolar[9,10] long-range interactions, but without a magnetic field. Here, we examine the effects of a frustrating long-range interaction on the phase diagram of a one-dimensional, Ising model at zero temperature *in the presence of a magnetic field*. This work is motivated by the proposal of Spivak and Kivelson[11] (and generalized with Jamei *et al.*[12]) that the putative first order phase transition between the Wigner crystal and Fermi liquid phases of the interacting electron gas in two dimensions at $T=0$ is preempted, due to the long-range Coulomb interaction, by a series of "microemulsion" phases characterized by phase separation on a mesoscopic scale. In our system, the magnetic field rather than the density will tune the system between phases. In general, a system with a long-range interaction that frustrates the order favored by a short-range interaction will not macroscopically separate into the phases of the unfrustrated system once the long-range interaction is strong enough. This is because the short-range interaction energy increase due to having mesoscopic domains is smaller than the long-range interaction energy increase due to having macroscopic domains. (Thus, in such systems, the Maxwell construction for determining phase separation must be generalized.[13])

In this work, we demonstrate that a Coulomb frustrated Ising model in a spatial continuum in one dimension (and its generalization to other power laws) admits analytical solution at zero temperature. (Note that we use "Coulomb interaction" to refer to the three-dimensional case proportional to $1/r$.) We find that this system possesses a regime exhibiting modulated phases (i.e., mesoscopic phase separation), with a period that varies continuously with applied magnetic field.

II. MODEL

We study a one-dimensional frustrated Ising model given by adding to the familiar ferromagnetic Ising chain in a magnetic field, where only nearest neighbors interact, a competing long-range antiferromagnetic interaction with a power-law falloff. We investigate the model in a one-dimensional continuum, where its formal Hamiltonian is given by

$$\mathcal{H}_H = J\int_{-\infty}^{\infty} dr \left|\frac{d\sigma(r)}{dr}\right| - H\int_{-\infty}^{\infty} dr\sigma(r) + \frac{Q}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} drdr' v(\Delta r)\sigma(r)\sigma(r'), \quad (1)$$

where $\Delta r \equiv |r-r'|$. Here, $\sigma(r) = \pm 1$ is a bivalued function of position r (Ising spin), J and Q are parameters representing the strengths of the short-range ferromagnetic and long-range antiferromagnetic interactions, respectively, and H is the strength of a uniform magnetic field. In this paper, we take the long-range antiferromagnetic interaction as $v(r) = e^{-br}/(r+a)^{\lambda}$, with exponent $\lambda > 0$. Ultraviolet and infrared cutoffs a and b, respectively, must sometimes be retained to eliminate divergences. In the case $H=0$, this model has been solved by Giuliani *et al.*,[14] and for $H=0$ and $v(r)$ equal to the inverse Fourier transform of the inverse Laplacian, Grousson *et al.*[15] have studied the model in three dimensions under the assumption that the ground state is periodic.

It proves helpful to perform a Legendre transformation on the energy and work at fixed average spin density $\bar{\sigma} = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L}\sigma(x)dx$ instead of fixed magnetic field strength H, especially for $\lambda \leq 1$, where the interaction energy density is infrared divergent for $\bar{\sigma} \neq 0$. Since the field term in $\mathcal{H}_H$ is constant for fixed $\bar{\sigma}$, the formal Hamiltonian at a given fixed $\bar{\sigma}$ and $a, b \to 0$ is

$$\mathcal{H}_{\bar{\sigma}} = J\int_{-\infty}^{\infty} dr \left|\frac{d\sigma(r)}{dr}\right| + \frac{Q}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} drdr' \frac{\sigma(r)\sigma(r')}{|r-r'|^{\lambda}}. \quad (2)$$

III. DETAILS OF CALCULATION

To investigate the $T=0$ properties of this classical system, we minimize the energy density to find the ground state. We *assume* that the ground state has a simple periodic structure, where each period comprises a length $l_{\uparrow}$ of up spins followed by a length $l_{\downarrow}$ of down spins. In the $H=0$ case, it has been proved[14] that the ground state must be of this form, with $l_{\uparrow}=l_{\downarrow}$. In the Appendix, Monte Carlo results are presented that confirm a simple periodic configuration to be the ground state for $H\neq 0$. A period has total length $L\equiv l_{\uparrow}+l_{\downarrow}$, and simple algebra reveals that $l_{\uparrow}=(1+\bar{\sigma})L/2$ and $l_{\downarrow}=(1-\bar{\sigma})L/2$. Minimizing the energy density for a given $\bar{\sigma}$ under this assumption is equivalent to minimizing the energy density of a single period with respect to variation in L (we choose L, but any of the interdependent variables $\{l_{\uparrow}, l_{\downarrow}, L\}$ could be used). The function $\sigma(r)$ is specified by the two parameters, $\bar{\sigma}$ and L:

$$\sigma(r) = \begin{cases} +1, & 0 < (r \bmod L) < l_{\uparrow} \\ -1, & l_{\uparrow} < (r \bmod L) < L, \end{cases} \quad (3)$$

with $l_{\uparrow}$ as given above. Before writing down an explicit formula for the energy density, we must choose a zero of energy, and the appropriate choice depends on the value of λ.

A. Case I: $0<\lambda<1$

When $0<\lambda<1$, we choose the zero of energy to be a uniform spin density of value $\bar{\sigma}$. This is equivalent to placing the system in a background jellium of "spin charge," with density $-\bar{\sigma}$, and results in the replacement of σ with

$$\sigma'(r) \equiv \sigma(r) - \bar{\sigma} \quad (4)$$

in Eq. (2). The energy density of a configuration with average spin $\bar{\sigma}$ and total period L, after again using $\Delta r \equiv |r-r'|$, is given by

$$\epsilon_{IR}(\bar{\sigma},L) = \frac{4J}{L} + \frac{Q}{2L}\int_0^L \int_{-\infty}^{\infty} drdr' \frac{e^{-b\Delta r}\sigma'(r)\sigma'(r')}{(\Delta r + a)^{\lambda}}. \quad (5)$$

The first term is the energy of two domain walls per period divided by the period length L. Our zero of energy has been chosen to eliminate the infrared divergence, making b unnecessary. The ultraviolet cutoff a must be kept when $\lambda \geq 1$. Thus, if we restrict ourselves to $\lambda<1$, the energy density of a configuration with average spin $\bar{\sigma}$ and total period L is given by

$$\epsilon_{IR}(\bar{\sigma},L) = \frac{4J}{L} + \frac{Q}{2L}\int_{-0}^{L} \int_{-\infty}^{\infty} drdr' \frac{\sigma'(r)\sigma'(r')}{|r-r'|^{\lambda}} \quad (6)$$

and is ensured to be finite. The function σ' is periodic with period L, and we define its Fourier transform:

$$\sigma'(r) = \sum_G \sigma'_G e^{irG}, \quad (7)$$

$$\sigma'_G = \frac{1}{L}\int_0^L \sigma'(r) e^{-irG} dr, \quad (8)$$

where the sum is over reciprocal lattice vectors $G=2\pi m/L$ for $m \in \mathbb{Z}$. Taking the Fourier transform of the second term in Eq. (5) gives

$$\epsilon(\bar{\sigma},L) = \frac{4J}{L} + \frac{Q}{2}\sum_G v_G |\sigma'_G|^2. \quad (9)$$

We have used $\int_0^L dr e^{ir(G_1-G_2)} = L\delta_{G_1,G_2}$ and have defined the Fourier transform of $v(r)$, $v_G = \int_{-\infty}^{\infty} dr v(r) e^{-irG}$. In the case $\lambda<1$, we can compute v_G with $a, b=0$ so that $v(r)=1/|r|^{\lambda}$. We obtain

$$v_G = 2\int_0^{\infty} \frac{\cos(Gr)}{r^{\lambda}} dr = 2G^{\lambda-1}\Gamma(1-\lambda)\sin\left(\frac{\pi\lambda}{2}\right) \quad (\lambda<1), \quad (10)$$

where we have skipped intermediate steps in the integration. Looking back at Eq. (9), we next must calculate the Fourier series coefficients of $\sigma'(r)$. We achieve this by calculating the Fourier coefficients σ_G of $\sigma(r)$. Then, since $\sigma'_G = \sigma_G$ for $G\neq 0$, and $\sigma'_{G=0}=0$ by definition, we obtain σ'_G:

$$\sigma_G = \frac{1}{L}\int_{-l_{\uparrow}}^{0} dr(1)e^{-irG} + \int_0^{l_{\downarrow}} dr(-1)e^{-irG}$$

$$= \frac{i}{GL}(2 - e^{il_{\uparrow}G} - e^{-il_{\downarrow}G}). \quad (11)$$

By substituting $G=2\pi m/L$ and inserting Eqs. (10) and (11) into Eq. (9), we arrive at a final expression for the energy density:

$$\epsilon_{IR}(\bar{\sigma},L) = \frac{4J}{L} + \frac{Q2^{\lambda+1}\Gamma(3-\lambda)\sin\frac{\pi\lambda}{2}}{L^{\lambda-1}(1-\lambda)(2-\lambda)} C(\bar{\sigma},\lambda), \quad (12)$$

where we have defined

$$C(\bar{\sigma},\lambda) = \sum_{m=1}^{\infty} \frac{1-(-1)^m\cos(\pi m\bar{\sigma})}{(\pi m)^{3-\lambda}}, \quad (13)$$

which converges for $\lambda<2$. Finally, we solve $\frac{\partial\epsilon_{IR}}{\partial L}=0$ to obtain the length of the period (L) that minimizes the energy density subject to the specified value of $\bar{\sigma}$ (and under the assumption $0<\lambda<1$):

$$L_0 = \left((2-\lambda)\frac{J}{Q}\left[2^{\lambda-1}\Gamma(3-\lambda)\sin\frac{\pi\lambda}{2}C(\bar{\sigma},\lambda)\right]^{-1}\right)^{1/(2-\lambda)}. \quad (14)$$

This expression is well defined on a larger interval than $0<\lambda<1$, and we will return to this point in subsequent sections. Also, if this optimal period length were written as a function of magnetic field instead of $\bar{\sigma}$, it would diverge at two critical values $\pm H_c$ that mark the onset of domain formation (see Fig. 2 and further discussion in the following section).

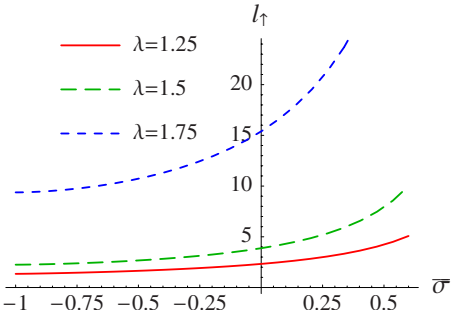

FIG. 1. (Color online) Size of the spin up domain in the ground state of Eq. (2) as a function of average spin $\bar{\sigma}$. The vertical axis is in units of $(\frac{2-\lambda}{\lambda}\frac{J}{Q})^{(1/2-\lambda)}$.

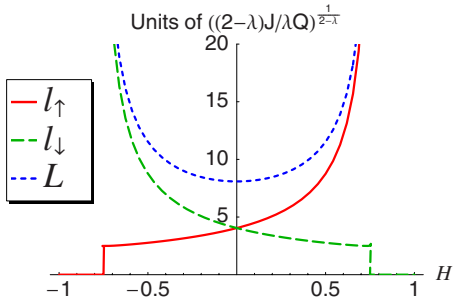

FIG. 2. (Color online) Size of the spin up domain $l_\uparrow$, spin down domain $l_\downarrow$, and their sum L (period of the configuration) that minimizes the energy of classical Hamiltonian (2) with λ=1.5. We obtain these quantities as a function of applied field H by performing a Legendre transformation.

B. Case II: $1<\lambda<2$

When $\lambda>1$, it is better to choose the fully polarized (ferromagnetic) state as the zero of energy in order to eliminate the *ultraviolet* divergence of the energy integrals. The energy density then reads

$$\epsilon_{UV}(\bar{\sigma},L)=\frac{4J}{L}+\frac{Q}{2L}\int_0^L\int_{-\infty}^{\infty}drdr'v(\Delta r)[\sigma(r)\sigma(r')-1]. \tag{15}$$

The first term is unchanged from the previous case, since the derivative is only sensitive to domain boundaries, and in the second term, we explicitly subtract the energy of a fully polarized configuration. In the interaction $v(r)$, we can set $a=0$ at any λ, since our choice of the zero of energy removes the ultraviolet divergence. The infrared cutoff b can be set to zero only when $\lambda>1$, and this determines the lowest λ where our final formula will be applicable. Performing a Fourier transform on the second term of Eq. (15) leads to

$$\epsilon_{UV}(\bar{\sigma},L)=\frac{4J}{L}+\frac{Q}{2}\sum_{G\neq 0}(v_G-v_0)|\sigma_G|^2. \tag{16}$$

Since σ_G is again given by Eq. (11), all that remains is to calculate the expression

$$v_G-v_0=2\int_0^{\infty}\frac{[\cos(Gr)-1]}{r^{\lambda}}dr$$

$$=2G^{\lambda-1}\Gamma(1-\lambda)\sin\left(\frac{\pi\lambda}{2}\right),\quad \lambda\in(1,3), \tag{17}$$

which converges to the second line when $1<\lambda<3$. Note that Eq. (17) is precisely the same as Eq. (10), which is valid for $0<\lambda<1$. Since σ_G agrees with σ'_G except at $G=0$, and $G=0$ is excluded in the sum of Eq. (16), the results obtained for $0<\lambda<1$ and $1<\lambda<2$ are given by the same expression (14). [The upper bound $\lambda<2$ is imposed by the sum (13).] We will show in the next section that this expression also holds for $\lambda=1$, extending its domain to $0<\lambda<2$.

In Fig. 1, the size of the up spin domain ($l_\uparrow$) is plotted at several values of $\lambda\in(1,2)$ as a function of $\bar{\sigma}$. As λ increases, the frustrating interaction becomes shorter ranged and the size of the domains becomes larger as expected. One must be careful, however, in interpreting Fig. 1 since the unit itself depends on λ in a way that tends to zero as $\lambda\to 2$. This exposes a difficulty of working with the continuum model with a variable exponent λ, as the length scale is dependent both on λ and J/Q. Figure 2 shows for fixed $\lambda=3/2$ the size of both spin domains, as well as their sum L, as a function of the applied field H (obtained via Legendre transform). We see an abrupt jump from zero to a finite domain size at $H=\pm H_c$, where there exists exactly one down or up domain, respectively, in a background of opposite spin.

C. Case III: $\lambda=1$

When $\lambda=1$, a and b cannot both be set to zero simultaneously in ϵ_{IR} or ϵ_{UV}, as one would encounter either an ultraviolet or infrared divergence, respectively. If $a,b>0$, then ϵ_{IR} and ϵ_{UV} are finite for all λ and differ only by an L-independent constant (a comparison of equations (9) and (16) show this constant to be $\frac{Q}{2}v_0(1-\bar{\sigma}^2)$). When $a>0$ and $b=0$, ϵ_{IR} is finite for all λ and ϵ_{UV} differs by an infinite L-independent constant. The reverse is true, with ϵ_{UV} finite, for $a=0$ and $b>0$. Thus, the optimal period lengths found by minimizing ϵ_{IR} and ϵ_{UV} agree and are defined for all λ so long as an appropriate cutoff is kept nonzero. When $0<\lambda<1$, we can take the remaining cutoff to zero in ϵ_{IR}, whereas when $1<\lambda<2$, we can do the same in ϵ_{UV}. In both cases, the result of minimizing the energy density leads to expression (14), written $L_0(\lambda)$ to emphasize its dependence on λ. Although it is tempting to infer that the optimal periodic length at $\lambda=1$ is $L_0(1)$, we must take care that the limits we have taken commute. The value $L_0(1)$ is obtained by taking the cutoff to zero, then λ to 1. We must check that the same result is given by first taking λ to 1, then the cutoff to zero.

We start with Eq. (15), which gives the energy density for all λ when $a=0$ and $b>0$. After first setting $\lambda=1$, we calculate the optimal period length in terms of the cutoff $b>0$ and *then* take the limit $b\to 0$. The Fourier transform of the long-range interaction $v(r)$ when $\lambda=1$, $a=0$, and $b>0$ is

$$v_G-v_0=2\int_0^{\infty}\frac{e^{-br}[\cos(Gr)-1]}{r}dr=-\ln\left(1+\frac{G^2}{b^2}\right). \tag{18}$$

Inserting into Eq. (16) and using $G=2\pi m/L$ gives

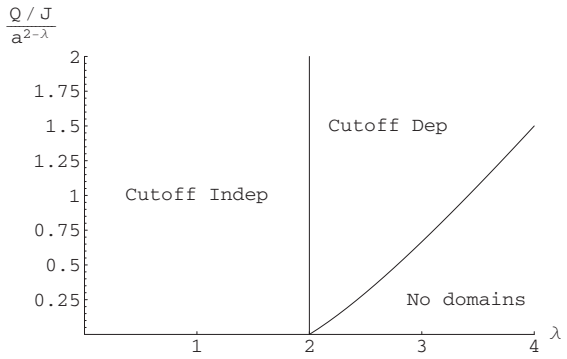

FIG. 3. Zero temperature phase diagram of Hamiltonian (2), showing where finite domains are present in the ground state and whether the domain sizes are dependent on the UV cutoff a in the limit $a \to 0$.

$$\epsilon_{UV,\lambda=1} = \frac{4J}{Q} - 2Q \sum_{m=-\infty}^{\infty} \ln\left[1 + \left(\frac{2\pi m}{bL}\right)^2\right] \times \left(\frac{1-(-1)^m \cos(\pi m \bar{\sigma})}{(\pi m)^2}\right). \tag{19}$$

Solving $\frac{\partial \epsilon_{UV,\lambda=1}}{\partial L} = 0$ yields an implicit equation for the optimal period length:

$$L_0' = \frac{J}{2Q}\left[\sum_{m=-\infty}^{\infty} h_m(L_0')\left(\frac{1-(-1)^m \cos(\pi m \bar{\sigma})}{(\pi m)^2}\right)\right]^{-1}, \tag{20}$$

where $h_m(L) = \left[1 + \left(\frac{bL}{2\pi m}\right)^2\right]^{-1}$. In the limit $b \to 0$, $h_m(L) \to 1$ and Eq. (20) reduces to the $\lambda = 1$ value of Eq. (14). Thus, we have shown that the result of taking the cutoff to zero and then λ to 1 is identical to taking λ to 1 and then the cutoff to zero. This implies that Eq. (14) is, indeed, valid at $\lambda = 1$ and gives a single continuous expression for the optimal periodic length when $0 < \lambda < 2$. We restate that result here for convenience:

$$L_0 = \left((2-\lambda)\frac{J}{Q}\left[2^{\lambda-1}\Gamma(3-\lambda)\sin\frac{\pi\lambda}{2} C(\bar{\sigma},\lambda)\right]^{-1}\right)^{1/(2-\lambda)}. \tag{21}$$

D. Case IV: $\lambda > 2$

When $\lambda > 2$, there is no problem with infrared divergence and b may be set to zero from the outset. To keep the energy density finite, the ultraviolet cutoff a, however, cannot be set to zero and must be carried through the calculation. Although the analysis leading to Eqs. (16) and (17) can be extended to $\lambda > 2$, the sum in Eq. (13) would diverge, and it is preferable to analyze the system entirely in real space, starting from Eq. (15). Only regions where $\sigma(r) \neq \sigma(r')$ [i.e., $\sigma(r)\sigma(r') = -1$] yield nonzero contributions in the second term, and after some algebra, for $L \gg a$, the energy density can be written as

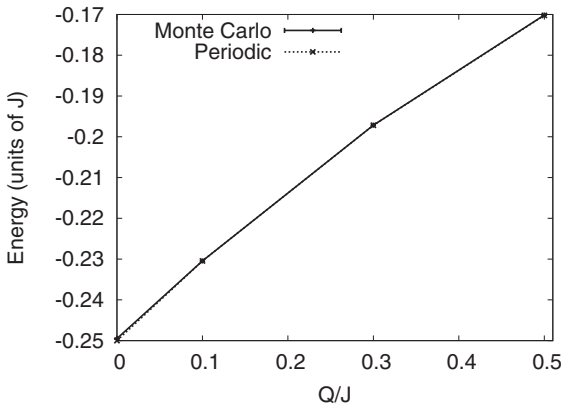

FIG. 4. Ground state energy density of the discrete Hamiltonian (A1) with $\lambda = 1.5$ on a chain with nearest neighbor spacing $a = 0.1$ and fixed magnetization $M = 0$. Monte Carlo results were obtained from systems of 2000 sites, converged after starting from a uniformly random initial configuration. The values plotted are the averages over many runs (each with a different starting configuration), and error bars show the standard deviation of the resulting distribution. The periodic result shows the minimum energy density that can be achieved by an exactly periodic configuration.

$$\epsilon_{UV}(\bar{\sigma},L) = \frac{4J}{L} + \frac{4Q}{(\lambda-2)(\lambda-1)L}\left[\frac{C'(\bar{\sigma},\lambda) + \alpha^{2-\lambda}}{L^{\lambda-2}} - \frac{1}{a^{\lambda-2}}\right], \tag{22}$$

where we have defined the following sum, convergent for $\lambda > 1$:

$$C'(\bar{\sigma},\lambda) = \sum_{n=1}^{\infty} [(n+\alpha)^{2-\lambda} - 2n^{2-\lambda} + (n-\alpha)^{2-\lambda}], \tag{23}$$

and $\alpha = (1+\bar{\sigma})/2$. For larger a/L, there are corrections to C' of order a/L. We set the derivative of the energy per site with respect to the period L to zero and, thus, find the ground state L:

$$\frac{d\epsilon_{UV}}{dL} = -\frac{4J}{L^2} - \frac{4Q}{(\lambda-2)L^{\lambda}}[C'(\bar{\sigma},\lambda) + \alpha^{2-\lambda}] + \frac{4Q}{(\lambda-1)(\lambda-2)L^2}a^{2-\lambda},$$

$$0 = -J + \frac{Q}{\lambda-2}\left[\frac{a^{2-\lambda}}{\lambda-1} - L^{2-\lambda}[C'(\bar{\sigma},\lambda) + \alpha^{2-\lambda}]\right],$$

$$\frac{J}{Q} = \frac{1}{\lambda-2}\left[\frac{a^{2-\lambda}}{\lambda-1} - \frac{\alpha^{2-\lambda}C'(\bar{\sigma},\lambda)}{L^{\lambda-2}}\right]. \tag{24}$$

In the present case of $\lambda > 2$, $C'(\bar{\sigma},\lambda) > 0$, so the last line above will have a solution with finite L only below a critical value of J/Q:

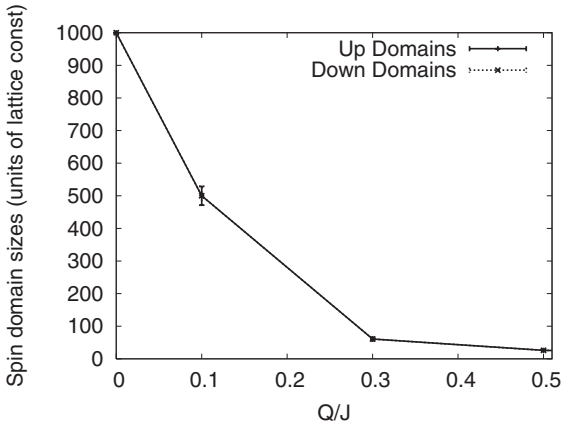

FIG. 5. Mean up and down spin domain sizes in the ground state of Hamiltonian (A1) with $\lambda=1.5$, lattice spacing $a=0.1$, and magnetization $M=0$. The ground state was found via Monte Carlo simulation on systems of 2000 sites. The sizes plotted are averages over many simulation runs (each with a different random starting configuration), and error bars show the standard deviation of the domain sizes.

$$\left(\frac{J}{Q}\right)_c = \frac{a^{2-\lambda}}{(\lambda-2)(\lambda-1)}. \tag{25}$$

For larger values of J/Q, the energy is minimized by infinite L, so the ground state has macroscopic phase separation, since the energy of a domain wall is then always positive.

Thus, for $\lambda>2$, this model with a given J/Q may or may not show finite domains in its ground state, depending on the cutoff a. In this regime, finite domains form in the ground state only for $J/Q<(J/Q)_c$, as specified by Eq. (25). Above this value, the ferromagnetic interaction is too strong relative to the antiferromagnetic interaction for domains to form. Figure 3 shows the regions of phase space where domains exist and where the domain size is cutoff independent in the limit $a\to 0$.

E. Case V: $\lambda=2$

In the special case $\lambda=2$, an analysis similar to the above $\lambda>2$ case can be done, which results in an energy density (valid for $L\gg a$)

$$\epsilon_{UV}(\bar{\sigma},L) = \frac{4J}{L} + \frac{4Q\beta^2}{L}\left[\ln\left(\frac{a}{\alpha L}\right) - \frac{\alpha L}{a}\right] + \frac{4Q\alpha^2}{L}\left[\ln\left(\frac{a}{\beta L}\right) - \frac{\beta L}{a}\right] + \frac{8Q\alpha\beta}{L}\ln\left(\frac{a}{\alpha\beta L}\right) + \frac{F(\bar{\sigma})}{L}, \tag{26}$$

where $\beta=(1-\bar{\sigma})/2$ and $F(\bar{\sigma})$ is a function independent of L. By solving $\frac{d\epsilon_{UV}}{dL}=0$, we find that the ground state period L_0'' is exponentially dependent on J/Q with a prefactor proportional to a:

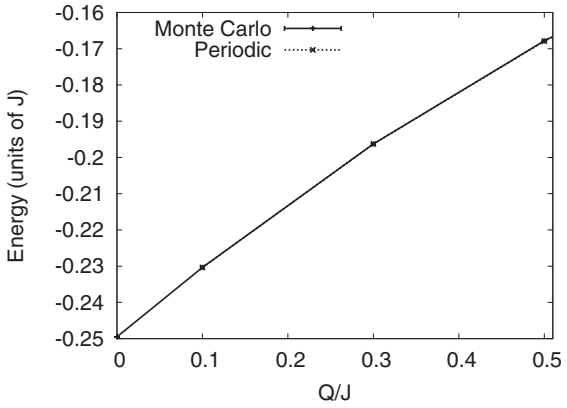

FIG. 6. Ground state energy density of Eq. (A1) with $\lambda=1.5$, $a=0.1$, and $M=0.3$. Monte Carlo and periodic configuration results obtained as for Fig. 4.

$$L_0'' = a\exp\left[\frac{J}{Q} + F'(\bar{\sigma})\right], \tag{27}$$

where $F'(\bar{\sigma})$ is another function independent of L and of order unity for $\bar{\sigma}\in[-1,+1]$. This form can be argued from dimensional analysis, since J/Q is dimensionless at $\lambda=2$, and the only length scale in the problem is a. The boundary case $\lambda=2$ separates the regime where L approaches a finite limit given by Eq. (14) as $a\to 0$ for $\lambda<2$ from the cutoff-dependent regime for $\lambda>2$.

IV. CONCLUDING REMARKS

In summary, we have studied the generalized Coulomb-frustrated Ising model in a one-dimensional continuum for different exponents λ of the long-range interaction ($\lambda=1$ for the three-dimensional Coulomb case). We have derived an analytic solution for the ground state domain configuration of this model under the assumption that the ground state configuration has a simple periodic structure. This assumption has been confirmed by Monte Carlo simulations (see the Appendix). Such simulations can be done in any dimension d, and mark an avenue for future work; more complicated domain patterns presumably do occur for $d>1$.

We find that for $0<\lambda<2$, as the magnetic field H is increased from $-\infty$, the ground state is ferromagnetic until a critical field $-H_c$ is reached, at which point a single finite-length domain of flipped spins is formed in an otherwise polarized background. For $-H_c<H<H_c$, periodic configurations with $l_\uparrow$ up spins followed by $l_\downarrow$ down spins become the ground state, and the system is said to be in a microemulsion phase. In higher dimensions, much numerical work has been done in the absence of a magnetic field (see Refs. 7–9), and applications have been made to magnetic thin films[16] in two dimensions as well as the metal-insulator transition in two and three dimensions.[17] Analytical expressions analogous to our results can be derived assuming that the ground state is simply periodic along only one direction[10] (i.e., a stripe configuration). However, in higher dimensions, the ground state is not necessarily a stripe configuration. In fact, for suffi-

ciently strong magnetic field, the ground state is believed to be a bubble configuration, in which droplets of the minority phase exist in a sea of the majority phase.[18] More generally, one is forced to resort to approximations and numerics. The numerical Monte Carlo work can be readily extended to higher dimensions and can be used to investigate domain formation and behavior as a magnetic field is varied (i.e., in the non-charge-neutral case).

Our solutions describe how $l_\uparrow$ increases and $l_\downarrow$ decreases with increasing H. At zero field, $l_\uparrow = l_\downarrow$, and at $H = \pm H_c$, we find that $l_\uparrow$ or $l_\downarrow$ diverges to infinity, respectively, while the length of the minority domain remains nonzero and finite. For $\lambda \geqslant 2$, whether or not the microemulsion phases appear in the transition between fully polarized up and down states depends on the dimensionless quantity $(Qa^{\lambda-2})/J$. In the case that microemulsion phases do occur in this region, their properties depend on the ultraviolet cutoff a. Our results provide more explicit examples of models with frustrating and sufficiently long-range interactions that have ordered microemulsion phases instead of macroscopic phase separation. However, we do not expect this long-range order to survive for any λ at finite temperature, since the system is one-dimensional and there exists a soft mode that does not couple to the long-range interaction (in which the domain walls move in a way that preserves the overall magnetization). Also, whether such long-range order is obtained in an inherently *quantum* system such as the two-dimensional electron gas requires further investigation. In this context, it is worth noting that claims exist in the literature in favor of the classical scenario, both for the electron gas[11] and for the highly disordered Anderson model with long-range Coulomb interactions.[19]

APPENDIX: MONTE CARLO ANALYSIS

The approximation central to this paper is that the ground state of $\mathcal{H}_{\bar{\sigma}}$ [Eq. (2)] has a simple periodic structure. While this has been proved for $H=0$ (i.e., $\bar{\sigma}=0$), no such result exists for nonzero magnetization. Thus, to justify the approximation, we have performed Monte Carlo simulations on a discretized version of Eq. (2):

$$\mathcal{H}_{\bar{\sigma}} = -J\sum_i \sigma_i\sigma_{i+1} + Q\sum_{ij}\frac{\sigma_i\sigma_j}{|i-j|^\lambda}. \tag{A1}$$

If we introduce a lattice spacing parameter a, then in the limit $J/Q \to 0$ and $a \to 0$ such that $\frac{Qa^{\lambda-2}}{J} = C$ for constant C, the above Hamiltonian is equivalent to the continuum formulation (2) with $J/Q = C$. We solve for the ground state the discretized model using a Monte Carlo algorithm with simulated annealing. Updates are governed by a Wolff cluster method,[20] which preserves the overall magnetization and, thus, $\bar{\sigma}$ is fixed during a simulation run (similar to the analytical calculation). Enough runs are averaged over so that the variation in the sizes of the resulting domains is negligible. Figure 4 is a typical comparison of the final (converged) Monte Carlo energy and the optimal energy of a periodic configuration as a function of Q/J for fixed exponent $\lambda = 1.5$, magnetization $M=0$, and lattice spacing $a=0.1$. We generally see a nearly exact overlap of the energies. Furthermore, we can look at sizes of the spin up and spin down domains, and find that the width of these distributions is very narrow. This is shown in Fig. 5, where we plot for the same system the mean domain size as a function of Q/J and use error bars to indicate the standard deviation of the same quantity.

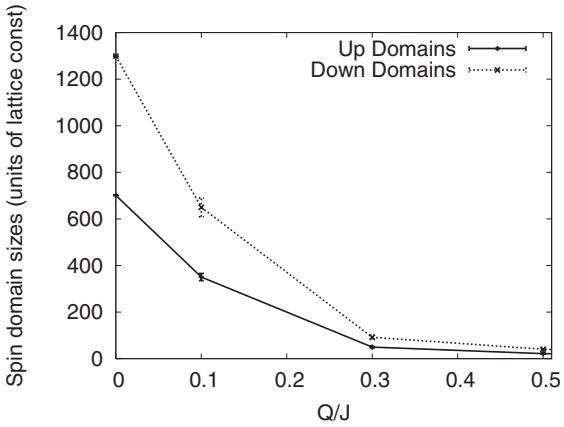

FIG. 7. Mean up and down spin domain sizes in the ground state of Hamiltonian (A1) with $\lambda = 1.5$, lattice spacing $a=0.1$, and magnetization $M=0.3$. The ground state was found via Monte Carlo simulation as in Fig. 5.

The conclusion that the ground state is a periodic alternation of uniformly sized domains for $M=0$ only confirms the aforementioned proof.[14] However, analogous results at $M \neq 0$, shown for $M=0.3$ in Figs. 6 and 7, strongly support the assertion that this conclusion is true for any magnetization. In particular, we find universally that spin configuration converged upon by each Monte Carlo run has an energy density which closely approaches, but is *greater* than, that of the optimal exactly periodic configuration.

In summary, we find that at any $\bar{\sigma} \in [-1,1]$, and for all values of J/Q, the ground state consists of uniform spin up domains of length $l_\uparrow$ interleaved with uniform spin down domains of length $l_\downarrow$ (but $l_\uparrow \neq l_\downarrow$ when $M \neq 0$). Thus, these results give convincing numerical support for the central assumption made in this paper.

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